

Optimal Exact Repair Strategy for the Parity Nodes of the $(k + 2, k)$ Zigzag Code

Jie Li and Xiaohu Tang, *Member, IEEE*

Abstract

In this paper, we reinterpret the $(k + 2, k)$ Zigzag code in coding matrix and then propose an optimal exact repair strategy for its parity nodes, whose repair disk I/O approaches a lower bound derived in this paper.

Index Terms

Distributed storage, MSR code, optimal repair, Zigzag code.

I. INTRODUCTION

Distributed storage systems built on huge numbers of storage nodes have wide applications in peer-to-peer storage systems such as OceanStore [12], Total Recall [1] and DHash++ [5]. Erasure code, which can provide both protection against node failures and efficient data storage, is very common in distributed storage systems [2], [3], [4], [8], [18], [19]. For instance, as a special class of erasure code, RAID-6 is a popular scheme for tolerating any two node failures [11].

Upon failure of a single node, a self-sustaining system should repair the failed node in order to retain the same redundancy. In the literature, there are mainly two repair types: exact repair and functional repair. Compared with the latter, exact repair is preferred since it does not incur additional significant system overhead by regenerating the exact replicas of the lost data at the failed node [7]. Generally speaking, there are several metrics to evaluate the performance of node repair, such as the *repair bandwidth*, which is defined as the amount of data downloaded from surviving nodes to repair a failed node, the *disk I/O*, which is defined as the amount of data read.

Recently, Dimakis *et al.* [6] introduced a new class of erasure code for distributed storage systems named *minimum storage regenerating* (MSR) code. The distributed storage system deploys a $(k + r, k)$ MSR code to store a file of size $M = kN$ symbols across n nodes, each node keeping N symbols. The $(k + r, k)$ MSR code has the *optimal repair property* that the repair bandwidth $\gamma = \frac{d}{(d-k+1)}N$ is minimal, which is achieved by downloading $\frac{N}{d-k+1}$ symbols from each of any $k \leq d \leq k + r - 1$ surviving nodes when repairing a failed node. In this paper, we only focus on the exact repair of high rate MSR codes. When $r = 1$, the repair bandwidth is the highest, i.e., $\gamma = M$. When $r = 2$ and $d = k + 1$, MSR code is very desirable since it can achieve the highest rate $\frac{k}{k+2}$ for $\gamma = (k + 1)N/2 < M$. In addition, $(k + 2, k)$ MSR code can be alternative to RAID-6 schemes.

So far, several explicit constructions of $(k + 2, k)$ MSR codes have been presented [9], [10], [13], [15]. Among them, the $(k + 2, k)$ Zigzag code in [13], which is defined by a series of permutations, is of great interest because of:

- (i) Optimal update disk I/O property (also known as optimal update property in [13]) that only itself and one symbol at each parity node need an update when a symbol in a systematic node is rewritten;
- (ii) Optimal repair disk I/O property (also known as optimal rebuilding in [13]) for systematic nodes that the repair disk I/O of a systematic node is equal to the minimal repair bandwidth;
- (iii) Small alphabet size of 3 so that it can be easily implemented;
- (iv) The storage $N = 2^{k-1}$ achieves the theoretic lower bound on the storage per node for $(k + 2, k)$ MSR codes with both optimal update disk I/O and optimal repair disk I/O for systematic nodes [13].

However, the parity nodes of the $(k + 2, k)$ Zigzag code was trivially repaired by downloading all the original data in [13], i.e., the download bandwidth reaches the maximal value $\gamma = M$. In order to acquire the optimal repair property for both systematic nodes and parity nodes, a $(k, k - 2)$ MSR code was presented in [16] based on a modification of the $(k + 2, k)$ Zigzag code, but at cost of sacrificing two systematic nodes while maintaining the same storage per node $N = 2^{k-1}$. It should be noted that only the $(k + 2, k)$ Hadamard MSR code in [10] shares the optimally repair property of all the nodes in the all aforementioned codes.

In this paper, without changing the original structure of the $(k + 2, k)$ Zigzag code, we propose an optimal repair strategy for the two parity nodes, whose download bandwidth achieves the minimal value $\gamma = (k + 1)N/2$. A comparison of the properties of various known $(k + 2, k)$ MSR codes, such as the Zigzag code employing our repair strategy, the original Zigzag

J. Li is with the Information Security and National Computing Grid Laboratory, Southwest Jiaotong University, Chengdu, 610031, China (e-mail: jieli873@gmail.com).

X.H. Tang is with the Information Security and National Computing Grid Laboratory, Southwest Jiaotong University, Chengdu 610031, China, and also with the Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing 100048, China (e-mail: xhutang@swjtu.edu.cn).

code [13], the modified Zigzag code [16], and Hadamard code [10], is given in Table I. It is seen that the new repair strategy does not lose any good properties of the original Zigzag code, for examples, the optimal update disk I/O property, the optimal repair disk I/O property for systematic nodes, small alphabet size of 3, and so on. In contrast to the modified Zigzag code and Hadamard code with the same optimal repair property of all nodes, the Zigzag code employing the new repair strategy shows a clear advantage over the storage per node. Although the repair disk I/O of the parity node is not optimal, which is $kN + N - k$, larger than the minimal repair bandwidth $(k + 1)N/2$, it indeed approaches a lower bound on the disk I/O of Zigzag code given in this paper.

TABLE I
COMPARISON OF THE PROPERTIES OF SOME $(k + 2, k)$ MSR CODES WHERE q AND N DENOTE THE SIZE OF THE FINITE FIELD REQUIRED AND THE STORAGE PER NODE, RESPECTIVELY.

	q	N	Optimal Update Disk I/O	Optimal Repair Disk I/O		Optimal Repair	
				Systematic Nodes	Parity Nodes	Systematic Nodes	Parity Nodes
Zizag Code Employing New Repair Strategy	3	2^{k-1}	Yes	Yes	No	Yes	Yes
Original Zigzag Code [13]	3	2^{k-1}	Yes	Yes	No	Yes	No
Modified Zigzag Code [16]	3	2^{k+1}	No	Yes	Yes	Yes	Yes
Hadamard Code [10]	$2k + 3$	2^{k+1}	Yes	No	No	Yes	Yes

The rest of this paper is organized as follows. Section II introduces the structure of a $(k + 2, k)$ MSR code and the necessary and sufficient conditions for optimal repair of parity nodes. Section III proposes the $(k + 2, k)$ Zigzag code and reinterprets it in coding matrix. In Section IV, a lower bound on disk I/O to optimally repair the parity nodes of the $(k + 2, k)$ Zigzag code is presented. The optimal repair strategy for the parity nodes of the $(k + 2, k)$ Zigzag code is given in Section V.

II. OPTIMAL REPAIR FOR PARITY NODES OF $(k + 2, k)$ MSR CODES

Let q be a prime power and \mathbf{F}_q be the finite field with q elements. Assume that a file of size $M = kN$ is equally partitioned into k parts, respectively denoted by $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{k-1}$, where \mathbf{f}_j is a column vector of length N for $0 \leq j < k$. The file is encoded to a $(k + 2, k)$ MSR code and then stored across k systematic and two parity storage nodes, each node having storage N . The first k nodes are systematic nodes, which store the file parts $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{k-1}$ in an uncoded form respectively. Without loss of generality, assume that the two parity nodes, nodes k and $k + 1$, respectively store $\mathbf{f}_k = \mathbf{f}_0 + \mathbf{f}_1 + \dots + \mathbf{f}_{k-1}$ and $\mathbf{f}_{k+1} = A_0\mathbf{f}_0 + A_1\mathbf{f}_1 + \dots + A_{k-1}\mathbf{f}_{k-1}$ for some $N \times N$ matrices A_0, \dots, A_{k-1} over \mathbf{F}_q , where the matrix A_j is called the *coding matrix* for systematic node j , $0 \leq j < k$. To guarantee the MDS property, it is required that [10], [14]

$$\text{rank}(A_i) = \text{rank}(A_i - A_j) = N, 0 \leq i \neq j < k. \quad (1)$$

Table I illustrates the structure of a $(k + 2, k)$ MSR code.

TABLE II
STRUCTURE OF A $(k + 2, k)$ MSR CODE

Node 0	Node 1	...	Node $k - 1$	Node k	Node $k + 1$
\mathbf{f}_0	\mathbf{f}_1	...	\mathbf{f}_{k-1}	$\mathbf{f}_k = \sum_{i=0}^{k-1} \mathbf{f}_i$	$\mathbf{f}_{k+1} = \sum_{i=0}^{k-1} A_i \mathbf{f}_i$

When repairing a failed node j , the optimal repair property demands to download half data from each surviving node l , $0 \leq l \neq j < k + 2$, by multiplying its original data \mathbf{f}_l with an $N/2 \times N$ matrix of rank $N/2$, called *repair matrix*. In what follows, we review the requirement on repair matrices for the optimal repair of parity nodes of a $(k + 2, k)$ MSR code [10], [14].

Upon failure of the first parity node (node k), respectively downloading $S_a \mathbf{f}_j$ and $\tilde{S}_a \mathbf{f}_{k+1}$, $0 \leq j < k$, where S_a and \tilde{S}_a are two $N/2 \times N$ repair matrices of rank $N/2$, eventually one gets the following system of linear equations

$$\begin{pmatrix} S_a \mathbf{f}_0 \\ \tilde{S}_a \mathbf{f}_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} S_a \\ \tilde{S}_a A_0 \end{pmatrix} \mathbf{f}_k}_{\text{useful data}} - \sum_{l=1}^{k-1} \underbrace{\begin{pmatrix} S_a \\ \tilde{S}_a (A_0 - A_l) \end{pmatrix} \mathbf{f}_l}_{\text{interference by } \mathbf{f}_l}.$$

To cancel all the interference terms and then recover the target data \mathbf{f}_k , the optimal repair requires [10], [14]

$$\text{rank} \left(\begin{pmatrix} S_a \\ \tilde{S}_a A_0 \end{pmatrix} \right) = N \quad (2)$$

and

$$\text{rank} \left(\begin{pmatrix} S_a \\ \tilde{S}_a(A_0 - A_l) \end{pmatrix} \right) = \frac{N}{2}, \quad 1 \leq l < k. \quad (3)$$

Clearly, the disk I/O to optimally repair the first parity node is $kN_1 + N_2$ where N_1 and N_2 denote the nonzero columns of S_a and \tilde{S}_a respectively.

To repair the second parity node (node $k+1$), downloading $(S_b A_j) \mathbf{f}_j$ and $\tilde{S}_b \mathbf{f}_k$, $0 \leq j < k$, where S_b and \tilde{S}_b are two $N/2 \times N$ matrices of rank $N/2$, one obtains the following system of linear equations

$$\begin{pmatrix} S_b A_0 \mathbf{f}_0 \\ \tilde{S}_b \mathbf{f}_k \end{pmatrix} = \underbrace{\begin{pmatrix} S_b \\ \tilde{S}_b A_0^{-1} \end{pmatrix} \mathbf{f}_{k+1}}_{\text{useful data}} - \underbrace{\sum_{l=1}^{k-1} \begin{pmatrix} S_b \\ \tilde{S}_b(A_0^{-1} - A_l^{-1}) \end{pmatrix} A_l \mathbf{f}_l}_{\text{interference by } \mathbf{f}_l}.$$

Similarly, optimal repair demands [10], [14]

$$\text{rank} \left(\begin{pmatrix} S_b \\ \tilde{S}_b A_0^{-1} \end{pmatrix} \right) = N \quad (4)$$

and

$$\text{rank} \left(\begin{pmatrix} S_b \\ \tilde{S}_b(A_0^{-1} - A_l^{-1}) \end{pmatrix} \right) = \frac{N}{2}, \quad 1 \leq l < k. \quad (5)$$

Accordingly, the disk I/O to optimally repair the second parity node is the total number of nonzero columns of \tilde{S}_b and $S_b A_i$, $0 \leq i < k$.

III. REINTERPRETATION OF $(k+2, k)$ ZIGZAG CODE IN CODING MATRIX

Throughout this paper, let $k \geq 2$ and $N = 2^{k-1}$. Given an integer $0 \leq i < N$, let (i_1, \dots, i_{k-1}) be its binary expansion, i.e., $i = \sum_{j=1}^{k-1} 2^{k-1-j} i_j$. For simplicity, we do not distinguish a nonnegative integer i and its binary expansion if the context is clear.

Let $\{e_j\}_{j=1}^{k-1}$ be the standard vector basis over \mathbf{F}_2 of dimension $k-1$, i.e.,

$$e_j = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{k-1}, \quad 1 \leq j < k$$

with only the j th entry being nonzero. By convenience, set e_0 to be the all-zero vector.

In [13], the $(k+2, k)$ Zigzag code is characterized by the following permutation $P_j : [0, N-1] \rightarrow [0, N-1]$

$$P_j(x) = x \oplus e_j = \begin{cases} (x_1, \dots, x_{k-1}), & j = 0 \\ (x_1, \dots, x_{j-1}, x_j \oplus 1, x_{j+1}, \dots, x_{k-1}), & 0 < j < k \end{cases}$$

where \oplus denotes the addition in \mathbf{F}_2 . Obviously,

$$P_j^{-1}(x) = x \oplus e_j = P_j(x), \quad 0 \leq j < k. \quad (6)$$

For any integer $0 \leq l < N$, define Z_l as $Z_l = \{(i, j) | i = P_j^{-1}(l), 0 \leq j < k\}$, i.e.,

$$Z_l = \{(i, j) | i = l \oplus e_j, 0 \leq j < k\}$$

by (6). The structure of the $(k+2, k)$ Zigzag code is depicted in Table II, where the first parity node stores $f_{i,k} = \sum_{j=0}^{k-1} f_{i,j}$

and the second parity node stores $f_{i,k+1} = \sum_{(i,j) \in Z_l} \beta_{i,j} f_{i,j}$, $0 \leq i < N$ and $0 \leq j < k$, $\beta_{i,j} = (-1)^{i \cdot \sum_{l=0}^j e_l}$, i.e.,

$$\beta_{i,j} = \begin{cases} 1, & \text{if } j = 0 \\ (-1)^{i_1 + \dots + i_j}, & \text{otherwise} \end{cases} \quad (7)$$

In the following, we reinterpret the data stored at the second parity node of the $(k+2, k)$ Zigzag code in the form of coding matrix so that we can use Equations (2)-(5) to check the optimality of our new repair matrices in the next section.

Given an integer $k \geq 2$, recursively define k matrices $A_0^{(k)}, \dots, A_{k-1}^{(k)}$ of order N over \mathbf{F}_3 as

$$A_0^{(k)} = I_{2^{k-1}}, \quad A_1^{(k)} = \begin{pmatrix} & -I_{2^{k-2}} \\ I_{2^{k-2}} & \end{pmatrix}, \quad A_j^{(k)} = \begin{pmatrix} A_{j-1}^{(k-1)} & \\ & -A_{j-1}^{(k-1)} \end{pmatrix} \quad \text{for } 2 \leq j < k \quad (8)$$

TABLE III
STRUCTURE OF THE $(k+2, k)$ ZIGZAG CODE

Node 0	...	Node $k-1$	Node k	Node $k+1$
$f_{0,0}$...	$f_{0,k-1}$	$f_{0,k} = \sum_{j=0}^{k-1} f_{0,j}$	$f_{0,k+1} = \sum_{(i,j) \in Z_0} \beta_{i,j} f_{i,j}$
$f_{1,0}$...	$f_{1,k-1}$	$f_{1,k} = \sum_{j=0}^{k-1} f_{1,j}$	$f_{1,k+1} = \sum_{(i,j) \in Z_1} \beta_{i,j} f_{i,j}$
\vdots	\vdots	\vdots	\vdots	\vdots
$f_{N-1,0}$...	$f_{N-1,k-1}$	$f_{N-1,k} = \sum_{j=0}^{k-1} f_{N-1,j}$	$f_{N-1,k+1} = \sum_{(i,j) \in Z_{N-1}} \beta_{i,j} f_{i,j}$

where

$$A_0^{(2)} = I_2, \quad A_1^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

First of all, the following properties of the matrices in (8) are obvious.

Property 1. For any $k \geq 2$, the matrix $A_j^{(k)}$ in (8) with $1 \leq j < k$ satisfies

- (i) $(A_j^{(k)})^2 = -I_{2^{k-1}}$;
- (ii) Both each row and each column of $A_j^{(k)}$ have only one nonzero entry.

Next, we show that the matrix $A_j^{(k)}$ in (8) is just the coding matrix for systematic node j of the $(k+2, k)$ Zigzag code for all $0 \leq j < k$.

Theorem 1. The coding matrices of the $(k+2, k)$ Zigzag code are $A_0^{(k)}, \dots, A_{k-1}^{(k)}$, i.e.,

$$\mathbf{f}_{k+1} = A_0^{(k)} \mathbf{f}_0 + \dots + A_{k-1}^{(k)} \mathbf{f}_{k-1}$$

where $\mathbf{f}_j = (f_{0,j}, \dots, f_{N-1,j})^T$.

Proof: Let $A(l, i)$ denote the entry at row l and column i of matrix A . By Property 1-(ii), equations (6) and (7), it suffices to prove $A_j^{(k)}(l, P_j^{-1}(l)) = \beta_{P_j^{-1}(l), j}$, i.e.,

$$A_0^{(k)}(l, l) = A_0^{(k)}(l, l \oplus e_0) = \beta_{l,0} = 1, 0 \leq l < N \quad (9)$$

and

$$A_j^{(k)}(l, l \oplus e_j) = \beta_{l \oplus e_j, j} = (-1)^{l_1 + \dots + l_j + 1}, \quad 1 \leq j < k, 0 \leq l < N. \quad (10)$$

Obviously, (9) holds since $A_0^{(k)}$ is the identity matrix and (10) holds for $j = 1$, i.e., $A_1^{(k)}(l, l \oplus e_1) = (-1)^{l_1 + 1}, 0 \leq l < N$, by the definition in (8).

Hereafter, we prove (10) for $j \geq 2$ by the induction. Suppose that (10) holds for $k \geq 2$ and $1 \leq j < k$. Then,

$$\begin{aligned} & A_j^{(k+1)}(l, l \oplus e_j) \\ &= A_j^{(k+1)}((l_1, \dots, l_k), (l_1, \dots, l_{j-1}, l_j \oplus 1, l_{j+1}, \dots, l_k)) \\ &= (-1)^{l_1} A_{j-1}^{(k)}((l_2, \dots, l_k), (l_2, \dots, l_{j-1}, l_j \oplus 1, l_{j+1}, \dots, l_k)) \\ &= (-1)^{l_1 + \dots + l_j + 1} \end{aligned}$$

for $2 \leq j < k+1$ and $0 \leq l < 2^k$, where the last two equalities respectively follow from (8) and the assumption. ■

IV. BOUNDS ON DISK I/O TO OPTIMALLY REPAIR THE PARITY NODES OF THE ZIGZAG CODE

For a general $(k+2, k)$ MSR code over \mathbf{F}_q defined in Table I, Wang *et al.* [17] proved that the minimal disk I/O to repair the first and second parity nodes are respectively at least $(k+1)N/2$ and kN if $q = 2$. In fact, the assertion can be proved for $q > 2$ by almost the same proof in [17].

Specifically for the Zigzag code, in this section we give a more tight bound on the minimal disk I/O for the optimal repair of the parity nodes.

Firstly, we state a connection between the optimal repair strategies for the two parity nodes of the Zigzag code.

Lemma 1. If $S^{(k)}$ and $\tilde{S}^{(k)}$ are the repair matrices for the first parity node of the $(k+2, k)$ Zigzag code, then $\tilde{S}^{(k)} A_j^{(k)}, 0 \leq j < k$, and $S^{(k)}$ are the repair matrices for the second parity node, and vice versa.

Proof: Note from (1) and (8) that $A_0^{(k)} - A_l^{(k)} = I_N - A_l^{(k)}$ is nonsingular for $1 \leq l < k$. Then,

$$\begin{aligned}
\text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} \\ S^{(k)} (A_0^{(k)})^{-1} - (A_l^{(k)})^{-1} \end{pmatrix} \right) &= \text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} \\ S^{(k)} (I_N + A_l^{(k)}) \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} \\ S^{(k)} (I_N + A_l^{(k)}) \end{pmatrix} (I_N - A_l^{(k)}) \right) \\
&= \text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} (I_N - A_l^{(k)}) \\ S^{(k)} (I_N + A_l^{(k)}) (I_N - A_l^{(k)}) \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S^{(k)} \\ \tilde{S}^{(k)} (I_N - A_l^{(k)}) \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S^{(k)} \\ \tilde{S}^{(k)} (A_0^{(k)} - A_l^{(k)}) \end{pmatrix} \right) \tag{11}
\end{aligned}$$

where in the first and fourth identities we use Property 1-(i), i.e., $(A_l^{(k)})^2 = -I_N$ and then $(A_l^{(k)})^{-1} = -A_l^{(k)}$.

In addition,

$$\text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} \\ S^{(k)} (A_0^{(k)})^{-1} \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} \tilde{S}^{(k)} \\ S^{(k)} \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} S^{(k)} \\ \tilde{S}^{(k)} A_0^{(k)} \end{pmatrix} \right).$$

Therefore, the result can be obtained from (2), (3), (4) and (5). \blacksquare

Theorem 2. *The disk I/O to optimally repair the first or second parity node of the $(k+2, k)$ Zigzag code is at least $kN + \frac{k-3}{2(k-1)}N$.*

Proof: Suppose that $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ are two repair matrices for the first parity node of $(k+2, k)$ Zigzag code. According to the definition of repair disk I/O, we need to prove $kN_1 + N_2 \geq kN + \frac{k-3}{2(k-1)}N$, where N_1 and N_2 respectively denote the number of nonzero columns of the matrices $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$.

By (2) and (3), we have

$$\text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} A_0^{(k)} \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} \right) = N \tag{12}$$

and

$$\text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (A_0^{(k)} - A_l^{(k)}) \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (I_N - A_l^{(k)}) \end{pmatrix} \right) = \frac{N}{2}, \quad 1 \leq l < k. \tag{13}$$

For $0 \leq i < N$, denote by $S_a^{(k)}[i]$ and $\tilde{S}_a^{(k)}[i]$ the column i of $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$. Assume that columns $i_1, i_2, \dots, i_{N-N_1}$ of $S_a^{(k)}$ are zero columns. Note that in (13), $\text{rank}(S_a^{(k)}) = \text{rank}(\tilde{S}_a^{(k)}(I_N - A_l^{(k)})) = N/2$. Then, we have that $\tilde{S}_a^{(k)}(I_N - A_l^{(k)})[i_s] = \tilde{S}_a^{(k)}[i_s] - (\tilde{S}_a^{(k)} A_l^{(k)})[i_s]$ is also a zero column, i.e.,

$$(\tilde{S}_a^{(k)} A_l^{(k)})[i_s] = \tilde{S}_a^{(k)}[i_s] \quad \text{for } 1 \leq l < k \text{ and } 1 \leq s \leq N - N_1.$$

Further, it follows from Property 1-(ii) and (10) that only the $(i \oplus e_l)$ th entry in $A_l^{(k)}[i]$ is ± 1 , which implies $(\tilde{S}_a^{(k)} A_l^{(k)})[i_s] = \pm \tilde{S}_a^{(k)}[i_s \oplus e_l]$. Thus,

$$\tilde{S}_a^{(k)}[i_s \oplus e_l] = \pm \tilde{S}_a^{(k)}[i_s] \quad \text{for } 1 \leq l < k \text{ and } 1 \leq s \leq N - N_1. \tag{14}$$

On the other hand, it is seen from (12) that all the columns $i_1, i_2, \dots, i_{N-N_1}$ of $\tilde{S}_a^{(k)}$ are linearly independent, which indicates that

$$\{i_u \oplus e_l : 1 \leq l < k\} \cap \{i_v \oplus e_l : 1 \leq l < k\} = \emptyset \quad \text{for } 1 \leq u \neq v \leq N - N_1. \tag{15}$$

Therefore, applying (14) and (15) to $\text{rank}(\tilde{S}_a^{(k)}) = N/2$, we obtain $N/2 \leq N - (k-1)(N - N_1)$, i.e., $N_1 \geq N - \frac{N}{2(k-1)}$. By means of (11), we can prove $N_2 \geq N - \frac{N}{2(k-1)}$ in the same fashion. Hence,

$$kN_1 + N_2 \geq (k+1)(N - \frac{N}{2(k-1)}) = kN + N - \frac{N(k+1)}{2(k-1)} = kN + \frac{k-3}{2(k-1)}N.$$

That is, the assertion is valid for the first parity node.

For the second parity node of the $(k+2, k)$ Zigzag code, assume that $S_b^{(k)} A_j^{(k)}, 0 \leq j < k$, and $\tilde{S}_b^{(k)}$ are the repair matrices. According to the definition, the repair disk I/O is the total number of nonzero columns of the matrices $S_b^{(k)} A_j^{(k)}$

and $\tilde{S}_b^{(k)}, 0 \leq j < k$, which is $kN_1 + N_2$ by Property 1-(ii), where N_1 and N_2 respectively denote the number of nonzero columns of the matrices $S_b^{(k)}$ and $\tilde{S}_b^{(k)}$. By Lemma 1, it is known that $\tilde{S}_b^{(k)}$ and $S_b^{(k)}$ are two repair matrices for the first parity node. Therefore, by the analysis for the first parity node we have $N_1 \geq N - \frac{N}{2(k-1)}$ and $N_2 \geq N - \frac{N}{2(k-1)}$, i.e., $kN_1 + N_2 \geq kN + \frac{k-3}{2(k-1)}N$. ■

V. REPAIR MATRICES FOR THE PARITY NODES OF THE ZIGZAG CODE

In this section, we give the repair matrices for the parity nodes of the $(k+2, k)$ Zigzag code and verify that they satisfy (2), (3), (4) and (5).

Recursively define the $2^{k-2} \times 2^{k-1}$ matrices $E^{(k)}$ and $F^{(k)}$ over \mathbf{F}_3 as

$$E^{(k)} = \begin{pmatrix} E^{(k-1)} \\ F^{(k-1)} \end{pmatrix}, \quad F^{(k)} = \begin{pmatrix} F^{(k-1)} \\ E^{(k-1)} \end{pmatrix}, \quad k \geq 3 \quad (16)$$

where

$$E^{(2)} = \begin{pmatrix} 0 & -1 \end{pmatrix}, \quad F^{(2)} = \begin{pmatrix} -1 & 0 \end{pmatrix}. \quad (17)$$

Next recursively define the $2^{k-2} \times 2^{k-1}$ matrices $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ over \mathbf{F}_3 as

$$S_a^{(k)} = \begin{pmatrix} S_a^{(k-1)} & E^{(k-1)} \\ \tilde{S}_a^{(k-1)} & F^{(k-1)} \end{pmatrix}, \quad \tilde{S}_a^{(k)} = \begin{pmatrix} \tilde{S}_a^{(k-1)} & -F^{(k-1)} \\ S_a^{(k-1)} & E^{(k-1)} \end{pmatrix}, \quad k \geq 3 \quad (18)$$

where

$$S_a^{(2)} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \tilde{S}_a^{(2)} = \begin{pmatrix} 1 & 1 \end{pmatrix}. \quad (19)$$

Proposition 1. For $k \geq 2$, $\text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} A_0^{(k)} \end{pmatrix} \right) = N$.

Proof: When $k = 2$, the statement is easily checked. For any given $k \geq 2$, suppose that the statement is true. According to recursive definition in (18), we have

$$\begin{aligned} \text{rank} \left(\begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)} A_0^{(k+1)} \end{pmatrix} \right) &= \text{rank} \left(\begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)} \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} S_a^{(k)} & E^{(k)} \\ \tilde{S}_a^{(k)} & -F^{(k)} \\ S_a^{(k)} & E^{(k)} \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} S_a^{(k)} & E^{(k)} \\ \tilde{S}_a^{(k)} & -F^{(k)} \\ \tilde{S}_a^{(k)} & E^{(k)} \end{pmatrix} \right) \\ &= 2N \end{aligned}$$

since $\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} = \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} A_0^{(k)} \end{pmatrix}$ is an $N \times N$ matrix of full rank.

Thus, the proof is finished by the above induction. ■

Proposition 2. For $k \geq 2$, $\text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (A_0^{(k)} - A_1^{(k)}) \end{pmatrix} \right) = N/2$.

Proof: When $k = 2$, the statement is easily checked. When $k > 2$, by the recursive definitions in (8) and (18), we have

$$\begin{aligned} &\tilde{S}_a^{(k)} (A_0^{(k)} - A_1^{(k)}) \\ &= \tilde{S}_a^{(k)} (I_N - A_1^{(k)}) \\ &= \begin{pmatrix} \tilde{S}_a^{(k-1)} & -F^{(k-1)} \\ S_a^{(k-1)} \end{pmatrix} \left(\begin{pmatrix} I_{N/2} & \\ & I_{N/2} \end{pmatrix} - \begin{pmatrix} & -I_{N/2} \end{pmatrix} \right) \\ &= \begin{pmatrix} \tilde{S}_a^{(k-1)} + F^{(k-1)} & \tilde{S}_a^{(k-1)} - F^{(k-1)} \\ -S_a^{(k-1)} & S_a^{(k-1)} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)} (A_0^{(k)} - A_1^{(k)}) \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k-1)} & E^{(k-1)} \\ \tilde{S}_a^{(k-1)} + F^{(k-1)} & \tilde{S}_a^{(k-1)} - F^{(k-1)} \\ -S_a^{(k-1)} & S_a^{(k-1)} \end{pmatrix} \right) \\
&= \text{rank} \left(P \cdot \begin{pmatrix} S_a^{(k-1)} & E^{(k-1)} \\ \tilde{S}_a^{(k-1)} + F^{(k-1)} & \tilde{S}_a^{(k-1)} - F^{(k-1)} \\ -S_a^{(k-1)} & S_a^{(k-1)} \end{pmatrix} \cdot Q \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k-1)} + E^{(k-1)} & \\ \tilde{S}_a^{(k-1)} & \\ & \tilde{S}_a^{(k-1)} + F^{(k-1)} \\ & & S_a^{(k-1)} \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k-1)} + E^{(k-1)} \\ \tilde{S}_a^{(k-1)} \end{pmatrix} \right) + \text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k-1)} + F^{(k-1)} \\ S_a^{(k-1)} \end{pmatrix} \right) \tag{20}
\end{aligned}$$

where the two matrices P, Q are respectively defined by

$$P = \begin{pmatrix} I_{N/4} & & I_{N/4} \\ & I_{N/4} & \\ & -I_{N/4} & -I_{N/4} \\ & & I_{N/4} \end{pmatrix}, \quad Q = \begin{pmatrix} I_{N/2} & -I_{N/2} \\ I_{N/2} & \end{pmatrix}.$$

Next, we prove

$$\text{rank} \left(\begin{pmatrix} S_a^{(k)} + E^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k)} + F^{(k)} \\ S_a^{(k)} \end{pmatrix} \right) = N/2$$

for any $k \geq 2$ by the induction.

When $k = 2$, the statement is easily verified. For any $k \geq 2$, suppose that it is true. By the definition of $S_a^{(k+1)}$ and $\tilde{S}_a^{(k+1)}$ in (18), we then have

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_a^{(k+1)} + E^{(k+1)} \\ \tilde{S}_a^{(k+1)} \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k)} + E^{(k)} & E^{(k)} \\ \tilde{S}_a^{(k)} & \tilde{S}_a^{(k)} + F^{(k)} \\ & -F^{(k)} \\ & S_a^{(k)} \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} I_{N/2} & & I_{N/2} \\ & I_{N/2} & I_{N/2} \\ & I_{N/2} & \end{pmatrix} \begin{pmatrix} S_a^{(k)} + E^{(k)} & E^{(k)} \\ \tilde{S}_a^{(k)} & \tilde{S}_a^{(k)} + F^{(k)} \\ & -F^{(k)} \\ & S_a^{(k)} \end{pmatrix} \begin{pmatrix} I_N & -I_N \\ & I_N \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k)} + E^{(k)} \\ \tilde{S}_a^{(k)} \\ & \tilde{S}_a^{(k)} + F^{(k)} \\ & & S_a^{(k)} \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} S_a^{(k)} + E^{(k)} \\ \tilde{S}_a^{(k)} \end{pmatrix} \right) + \text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k)} + F^{(k)} \\ S_a^{(k)} \end{pmatrix} \right) \\
&= N
\end{aligned}$$

where the last identity comes from the assumption. Similarly, we can get $\text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k+1)} + F^{(k+1)} \\ S_a^{(k+1)} \end{pmatrix} \right) = N$. This completes the proof after substituted into (20). \blacksquare

Proposition 3. Given $k \geq 3$, $\text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)}(A_0^{(k)} - A_i^{(k)}) \end{pmatrix} \right) = N/2$ for all $2 \leq i < k$.

Proof: If $k = 3$, the statement is obvious. For any $k \geq 3$, assume that it is true for all $2 \leq j < k$. When $j \geq 2$, according to the definitions of $A_j^{(k+1)}$ in (8) and $S_a^{(k+1)}, \tilde{S}_a^{(k+1)}$ in (18),

$$\text{rank} \left(\begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)}(A_0^{(k+1)} - A_j^{(k+1)}) \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)}(I_{2N} - A_j^{(k+1)}) \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} U_j^{(k)} & W_j^{(k)} \\ V_j^{(k)} \end{pmatrix} \right)$$

for three $N \times N$ matrices

$$U_j^{(k)} = \begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)}(I_N - A_{j-1}^{(k)}) \end{pmatrix}, V_j^{(k)} = \begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)}(I_N + A_{j-1}^{(k)}) \end{pmatrix}, W_j^{(k)} = \begin{pmatrix} E^{(k)} \\ -F^{(k)}(I_N + A_{j-1}^{(k)}) \end{pmatrix},$$

by the recursive definitions which satisfy

$$W_j^{(k)} = \begin{cases} -U_j^{(k)} + R^{(k)}V_j^{(k)}, & \text{if } j = 2 \\ U_j^{(k)}Q^{(k)} - P^{(k)}V_j^{(k)}, & \text{if } j > 2 \end{cases}$$

where

$$R^{(k)} = \begin{pmatrix} \mathbf{0}_{N/4} & I_{N/4} & I_{N/4} & \mathbf{0}_{N/4} \\ -I_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & I_{N/4} \\ \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & I_{N/4} \\ \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & -I_{N/4} & \mathbf{0}_{N/4} \end{pmatrix}, P^{(k)} = \begin{pmatrix} \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} \\ I_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} \\ \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & \mathbf{0}_{N/4} \\ \mathbf{0}_{N/4} & \mathbf{0}_{N/4} & I_{N/4} & \mathbf{0}_{N/4} \end{pmatrix}, Q^{(k)} = \begin{pmatrix} \mathbf{0}_{N/2} & \mathbf{0}_{N/2} \\ I_{N/2} & \mathbf{0}_{N/2} \end{pmatrix}$$

and $\mathbf{0}_N$ denotes the zero matrix of order N .

Hence,

$$\begin{aligned} & \text{rank} \left(\begin{pmatrix} S_a^{(k+1)} \\ \tilde{S}_a^{(k+1)}(A_0^{(k+1)} - A_j^{(k+1)}) \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)}(I_N - A_{j-1}^{(k)}) \end{pmatrix} \right) + \text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)}(I_N + A_{j-1}^{(k)}) \end{pmatrix} \right) \end{aligned} \quad (21)$$

for $j \geq 2$.

Further, note from (1) that $A_0^{(k)} - A_{j-1}^{(k)} = I_N - A_{j-1}^{(k)}$ is nonsingular if $j \geq 2$. Then,

$$\begin{aligned} & \text{rank} \left(\begin{pmatrix} \tilde{S}_a^{(k)} \\ S_a^{(k)}(I_N + A_{j-1}^{(k)}) \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} S_a^{(k)} \\ \tilde{S}_a^{(k)}(I_N - A_{j-1}^{(k)}) \end{pmatrix} \right) \\ &= N/2 \end{aligned}$$

where in the first identity we use (11) and the in last identity we use the assumption if $j \geq 3$ and Proposition 2 if $j = 2$. This completes the proof after substituted into (21). \blacksquare

The following main result is immediate.

Theorem 3. $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ that defined by (16), (17), (18) and (19) are the repair matrices for the first parity node of the $(k+2, k)$ Zigzag code, whose repair disk I/O is $kN + N - k$.

Proof: The optimal repair property of repair matrices $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ is obvious from Propositions 1, 2 and 3.

Note that there is only one zero column in $S_a^{(k)}$ and no zero columns in $\tilde{S}_a^{(k)}$, which means $N-1$ elements should be read in each of the systematic nodes and all the N elements should be read in the second parity node to repair the first parity node. Thus the disk I/O to repair the first parity node is $kN + N - k$. \blacksquare

By Lemma 1, the second parity node of the $(k+2, k)$ Zigzag code can also be optimally repaired. However, if we use $S_b^{(k)}A_i^{(k)}, 0 \leq i < k$ and $\tilde{S}_b^{(k)}$ as the repair matrices, where $S_b^{(k)} = \tilde{S}_a^{(k)}$ and $\tilde{S}_b^{(k)} = S_a^{(k)}$ are defined by (16), (17), (18) and

(19), then its repair disk I/O will be $kN + N - 1$ since $S_a^{(k)}$ has only one zero column and $\tilde{S}_a^{(k)} A_i^{(k)}$ has no zero columns for $0 \leq i < k$. In the following, by choosing another initial values of $E^{(2)}$, $F^{(2)}$, $S_a^{(2)}$, $\tilde{S}_a^{(2)}$ in (17) and (19), the disk I/O to optimally repair the second parity node can also be reduced to $kN + N - k$.

Reset

$$E^{(2)} = \begin{pmatrix} -1 & 0 \end{pmatrix}, \quad F^{(2)} = \begin{pmatrix} 0 & -1 \end{pmatrix}, \quad S_a^{(2)} = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad \tilde{S}_a^{(2)} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad (22)$$

then we have the following result.

Theorem 4. Let $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ be defined by (22), (16) and (18), then $S_b^{(k)} A_i^{(k)}$, $0 \leq i < k$ and $\tilde{S}_b^{(k)}$ are the repair matrices for the second parity node of the $(k+2, k)$ Zigzag code where $S_b^{(k)} = \tilde{S}_a^{(k)}$ and $\tilde{S}_b^{(k)} = S_a^{(k)}$. Moreover, the disk I/O to optimally repair the second parity node is $kN + N - k$.

Proof: Firstly, it can be easily verified that the results in Propositions 1, 2 and 3 are also hold for $S_a^{(k)}$ and $\tilde{S}_a^{(k)}$ defined from the initial values $E^{(2)}$, $F^{(2)}$, $S_a^{(2)}$ and $\tilde{S}_a^{(2)}$ in (22). Secondly, it follows from Lemma 1 that $\tilde{S}_a^{(k)} A_i^{(k)}$, $0 \leq i < k$ and $S_a^{(k)}$ are the repair matrices for the second parity node of the $(k+2, k)$ Zigzag code. ■

From Theorems 3 and 4, it is seen that the disk I/O to optimally repair the parity nodes of the Zigzag code is very close to the lower bound given in Lemma 2.

Finally, we give some examples of the repair matrices for the parity nodes of the $(k+2, k)$ Zigzag code.

Example 1. The first parity node of the $(5, 3)$ Zigzag code, $(6, 4)$ Zigzag code, and $(7, 5)$ Zigzag code, can be respectively optimally repaired by the following matrices

$$\begin{aligned} S_a^{(3)} &= \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \tilde{S}_a^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ S_a^{(4)} &= \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_a^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \\ S_a^{(5)} &= \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \\ \tilde{S}_a^{(5)} &= \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The second parity node of the $(5, 3)$ Zigzag code, $(6, 4)$ Zigzag code, and $(7, 5)$ Zigzag code, can be respectively optimally repaired by the following matrices

$$\begin{aligned} S_b^{(3)} &= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \tilde{S}_b^{(3)} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ S_b^{(4)} &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_b^{(4)} = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
S_b^{(5)} &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \\
\tilde{S}_b^{(5)} &= \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

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